ENGINEERING PAPER 3090



# ANALYSIS OF COMPLEX STRUCTURES SUBJECTED TO RANDOM EXCITATION

5-9 OCTOBER 1964

BY
C. M. FULLER
AND
D. N. ROUDEBUSH
SATURN ACOUSTIC & STRUCTURAL DYNAMICS SECTION
MISSILE AND SPACE SYSTEMS DIVISION
DOUGLAS AIRCRAFT COMPANY, INC.

TO BE PRESENTED TO
SOCIETY OF AUTOMOTIVE ENGINEERS
NATIONAL AERONAUTICS & SPACE ENGINEERING MEETING
LOS ANGELES, CALIFORNIA
5-9 OCTOBER 1964

DOUGLAS

DOUGLAS MISSILE & SPACE SYSTEMS DIVISION

3/108 2

## ABSTRACT

A method is presented for the dyanamical analysis of complex structures subjected to a random environment. The excitations considered are random motion of the base and multiple random loadings. Equations are developed for computing the response using generalized parameters. Cross power spectral densities are used to account for the correlation of the multiple random inputs. The main advantage of this method is the capability of wide application.

An example is given illustrating the use of this method to determine the response to random motion of the base, an acoustic field and both excitations simultaneously.

## CREDIT

Phases of work presented herein were accomplished by the Missiles and Space Systems Division of Saturn contract NAS7-1 sponsored by the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration.

# TABLE OF CONTENTS



## APPENDICES

## LIST OF ILLUSTRATIONS



## TABLE



## LIST OF CHARTS

Page

# ANALYSIS OF COMPLEX STRUCTURES SUBJECTED TO RANDOM EXCITATION

BY

C. M. Fuller

and

D. N. Roudebush

## INTRODUCTION

The dynamic environment experienced by missile and space vehicles during static firing and flight are usually sufficiently severe to influence portions of the design. Because of the severity of this environment, an analytical method which will reliably predict the response to a random environment must be available. Besides being useful in demonstrating that the design is satisfactory, the analysis finds many applications in evaluating the data from vehicle flights and laboratory testing.

The analysis presented here, determines the response of a complex structural system by superimposing the response of several natural modes. This modal approach is useful because it retains the generality required for application to a wide variety of structures. This paper presents and discusses the equations to be evaluated.

An example is given which illustrates the application of the method for the cases where the excitation is random vibration of the base, an acoustic field, and the two excitations simultaneously.

## TRANSFER FUNCTIONS

The general differential equations of motion for a forced lumped parameter system may be written as given in reference 1.

$$\left[M\right]\left\{\ddot{x}\right\} + i\left[B\right]\left\{x\right\} + \left[K\right]\left\{x\right\} = \left\{F\right\} \qquad (1)$$

Where $\left[M\right]$, $\left[B\right]$, $\left[K\right]$, and $\left\{F\right\}$ are the sectional mass, damping, stiffness, and forcing matrices respectively.

Considering a solution of the form

$$X(y,t) = \left[\psi_n(y)\right] \left\{c\right\} e^{i\omega t} \tag{2}$$

for the homogeneous part of Equation (1) and substituting Equation (2) in Equation (1) yields

$$(-\omega^2 \left[M\right] + i\left[B\right] + \left[K\right]) \left[\psi\right] \left\{c\right\} e^{i\omega t} = \left\{F\right\} \tag{3}$$

Using the property of orthogonality with respect to a weighting function of eigenvectors to complete the similarity transformation yields

$$(-\omega^2 \left[\mu\right] + i\left[\beta\right] + \left[\kappa\right]) \left\{C\right\} e^{i\omega t} = \left\{\rho\right\} \tag{4}$$

where

$$\left[\mu\right] = \left[\widetilde{\psi}\right] \left[M\right]\left[\psi\right] \tag{5}$$

$$\left[\beta\right] = \left[\widetilde{\psi}\right] \left[B\right]\left[\psi\right] \tag{6}$$

$$\left[\kappa\right] = \left[\widetilde{\psi}\right] \left[K\right]\left[\psi\right] \tag{7}$$

$$\left\{\rho\right\} = \left[\widetilde{\psi}\right] \left\{F\right\} \tag{8}$$

The net effect of the similarity transformation is the diagonalization of the matrices, i.e., uncoupling of the matrices.

Let the forcing function take the form

$$\left\{\rho\right\} = A \left\{\rho'\right\} e^{i(\omega t + \theta_n)} \tag{9}$$

where

A is a scalar whose function is convenience when calculating transfer functions or total response.

$\theta_n$ is the phase angle to the reference phasor.

ıere

$x_b$ is the motion of the base

$x_i$ is the total motion of the $i^{th}$ degree of freedom

$\overline{x}_i$ is the relative motion of the $i^{th}$ degree with the base motion

:rforming the coordinate change, the equations of motion becomes

$$\left[M\right]\left\{\ddot{\overline{x}} + \ddot{x}_b\right\} + i\left[B\right]\left\{\overline{x}\right\} + \left[K\right]\left\{\overline{x}\right\} = 0 \tag{16}$$

·arranging Equation (16) gives

$$\left[M\right]\left\{\ddot{\overline{x}}\right\} + i\left[B\right]\left\{\overline{x}\right\} + \left[K\right]\left\{\overline{x}\right\} = -\left[M\right]\left\{\ddot{x}_b\right\} \tag{17}$$

The steady state portion of the forcing function, $A\left\{\bar{\rho}\right\}$, is found to be ven by

$$A\left\{\rho\right\} = -\ddot{x}_b\widetilde{\left[\psi\right]}\left[M\right]\left\{u\right\} \tag{18}$$

.ere

$\left\{u\right\}$ is a column matrix whose elements are the direction cosine between the force vector and the degree of freedom.

nce the total response is given by

$$H'(\omega) = -\ddot{x}_b\left[\gamma\right]\left(-\omega^2\left[\mu\right] + i\left[\beta\right] + \left[\kappa\right]\right)^{-1}\widetilde{\left[\psi\right]}\left[M\right]\left\{u\right\} \tag{19}$$

r a base input, $\ddot{x}_b$, of unity Equation (19) yields the transfer function, ω), for base excitations.

Application of the method described begins with the calculation of a mass trix $\left[M\right]$, a stiffness matrix $\left[K\right]$, or an influence coefficient matrix, $]^{-1}$.

Letting

$$\overline{\rho}_n = \rho_n' \ e^{i\theta_n} \tag{10}$$

then Equation (9) becomes

$$\{\rho\} = A \{\overline{\rho}\} e^{i\omega t} \tag{11}$$

and Equation (4) becomes

$$(-\omega^2 [\mu] + i [\beta] + [\kappa]) \{c\} e^{i\omega t} = A \{\overline{\rho}\} e^{i\omega t} \tag{12}$$

or

$$\{c\} = A (-\omega^2 [\mu] + i [\beta] + [\kappa])^{-1} \{\overline{\rho}\} \tag{13}$$

$c_n$ corresponds to the modal response. Transfer functions $H(\omega)$ or total response $H'(\omega)$ for all degrees of freedom are found by substituting Equation (13) into Equation (2) and retaining only the steady state portion of the solution. The transfer function differs from the total response in that for transfer functions the scalar A must be unity. This substitution yields

$$H'(\omega) = A_1 [\gamma] (-\omega^2 [\mu] + i[\beta] + [\kappa])^{-1} \{\overline{\rho}\} \tag{14}$$

where

> $\gamma_{mn}$ is the value of the desired response for a unit amplitude of the $n^{th}$ mode.

For the case where the input force is motion of the base, the equations of motion now must describe the motion relative to the base. Using a new coordinate systems, defined by Equation (15).

$$\overline{x}_i = x_i - x_b \tag{15}$$

The method of achieving the structural matrix depends on the complexity of the problem, ranging from hand calculations for simple structures to analysis of statically indeterminent structures by digital computers. In most practical applications the damping has little effect on the eigenvectors or eigenvalues and has been omitted in the calculation of eigenvalues and eigenvectors for the convenience of avoiding complex arithmetic.

The eigenvalues and eigenvectors for the equation

$$\left( \left[K\right]^{-1} \left[M\right] - \lambda \left[I\right] \right) = 0 \tag{20}$$

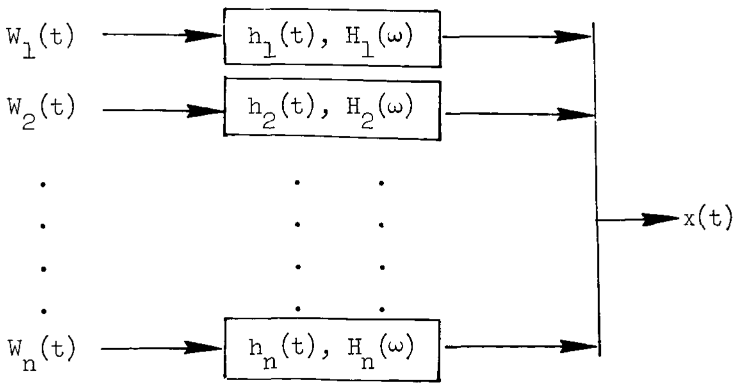
are computed. Using the eigenvectors as base vectors for a new coordinate system, the mass matrix is transformed to the generalized mass matrix, $\left[\mu\right]$. The generalized stiffness matrix, $\left[\kappa\right]$, is related to the generalized mass matrix by the following equation:

$$\left[\kappa\right] = \left[\omega^2_n\right]\left[\mu\right] \tag{21}$$

The transfer functions or total response is found by evaluating Equation (14) or (10). In the case that A is unity, the result is a transfer function and the type of response is determined by $\gamma_{mn}$.

## RESPONSE TO RANDOM ENVIRONMENTS

A linear multiple degree of freedom system with impulse response, $h_n(t)$, transfer functions, $H_n(\omega)$, and multiple random inputs, $W_n(t)$, can be represented schematically as follows:



The transfer function is defined in Reference 2 as

$$H(\omega) = F\left[P^{o}(t)\right] \;/\; F\left[P^{i}(t)\right] \tag{22}$$

where

$P^{i}(t)$ is the input function

$P^{o}(t)$ is the output function

$H(\omega)$ is the transfer function

$F$ denotes the Fourier transformation

If the input function is an impulse function such that the Fourier transform exists, then $H(\omega)$ is given by

$$H(\omega) = F\left[h(t)\right] \tag{23}$$

since

$$F\left[\delta(t)\right] = 1$$

The autocorrelation function is defined as

$$R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\; X(t)\; X(t+\tau) \tag{24}$$

From the convolution integral $X(t)$ and $X(t+\tau)$ are given by

$$X(t) = \sum \int_{-\infty}^{\infty} d\zeta\; W_j(t-\zeta)\; h_j(\zeta) \tag{25}$$

$$X(t+\tau) = \sum \int_{-\infty}^{\infty} d\eta\; W_k(t+\tau-\eta)\; h_k(\eta)$$

Combining Equations (24) and (25) yields

$$R_{jk}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \sum_{\tau} \int_{-\infty}^{\infty} d\zeta \; W_j(t - \zeta)\, h_j(\zeta)$$

$$\sum \int_{-\infty}^{\infty} d\eta \; W_k(t + \tau - \eta)\, h_k(\eta) \tag{26}$$

Rearranging the order of the integration of Equation (26) and noting that the correlation of the input is defined as

$$\overline{R}_{jk}(\tau - \eta + \zeta) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \; W_j(t - \zeta)\, W_k(t + \tau - \eta) \tag{27}$$

yields

$$R_{jk}(\tau) = \sum \sum \int_{-\infty}^{\infty} d\zeta \, h_j(\zeta) \int_{-\infty}^{\infty} d\eta \, h_k(\eta)\, \overline{R}_{jk}(\tau - \eta + \zeta) \tag{28}$$

Assuming that the Fourier transform of $R(\tau)$ exist, the power spectrum, $\phi(\omega)$ is given by the following expression

$$\phi(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} \sum \sum \int_{-\infty}^{\infty} d\zeta \, h_j(\zeta) \int_{-\infty}^{\infty} d\eta \, h_k(\eta)\, \overline{R}(\tau - \eta + \zeta) \tag{29}$$

Expressing $e^{-i\omega\tau}$ as

$$e^{-i\omega\tau} = e^{-i\omega(\tau + \zeta - \eta)} \; e^{-i\omega\eta} \; e^{i\omega\zeta} \tag{30}$$

and substituting into Equation (29) yields

$$\phi(\omega) = \sum\sum \int_{-\infty}^{\infty} d\zeta\, h_j(\zeta)\, e^{i\omega\zeta} \int_{-\infty}^{\infty} d\eta\, h_k(\eta)\, e^{-i\omega\eta}$$

$$\int_{-\infty}^{\infty} d\tau\, R_{jk}(\tau + \zeta - \eta)\, e^{-i\omega(\tau + \zeta - \eta)} \qquad (31)$$

$$\phi(\omega) = \sum\sum H_j(\omega)\, H_k^*(\omega)\, \bar{\phi}_{jk}(\omega) \qquad (32)$$

If $j \neq k$, then $\bar{\phi}_{jk}(\omega)$ is the cross power spectrum of the instantaneous values of the input time functions $W_j(t)$ and $W_k(t)$. For the case where $j = k$, $\phi_{jk}(\omega)$ is the power spectrum of the input.

Having determined the transfer functions for the system and knowing the power spectra and cross power spectra of the input, the power spectrum of the output is then found by evaluating Equation (32) for the desired frequency range.

## CONCLUDING REMARKS

The main feature of this method of analysis is its versatility. For the purpose of analysis, the required information consists of mass, damping, and stiffness matrices which may be obtained in a large variety of methods. If facilities are available, these parameters may be obtained experimentally.

Once the transfer functions have been computed, the choice of the type of excitation to which the structure is subjected and the type of response to be computed is nearly unrestricted.

This method has demonstrated its efficiency when used in conjunction with digital computers. The experienced user has the capability of increased accuracy by the use of more normal modes or increased efficiency by using only the most significant normal modes when computing the total response.

EXAMPLE

To illustrate the foregoing method a loaded plate, shown in figure 1,was used as an example. The excitations considered were random vibration of the base and an acoustic field impinging on the surface.

The influence coefficient matrix, for the lumped parameter model shown in figure 2, was calculated using the Douglas Aircraft Company Redundant Force Computer Program given in references 3 & 4 . Having formed the dynamical matrix from the mass matrix and influence coefficient matrix, the eigenvalues and eigenvectors for the dynamical equation were then calculated. The first nine natural frequencies for the 41 degree of freedom system are shown in Table 1.

Transfer functions were calculated assuming the damping for each mode to be 2% of the critical damping value. Charts 1, 2, and 3 show the transfer function for motion of mass $M_1$ normal to the plane of the plate, parallel to the width and parallel to the length respectively, when the excitation is motion of the base normal to the plane of the plate. Chart 4, 5, and 6 show functions of $M_1$ for normal incidence of the fluctuating pressure. The fluctuating pressure was defined to be perfectly correlated over the entire surface of the plate.

The responses of $M_1$ in the three axes of the plate for a 0.1 $g^2$/cps random input at the base are shown in Charts 7, 8, and 9. Charts 10, 11, and 12 show the response of $M_1$ to a 161 db acoustic field whose spectrum is such that the fluctuating pressure is .001 $(psi)^2$/cps over the frequency range of interest.

Charts 13, 14, and 15 show the response of $M_1$ in the respective directions to the two excitations acting simultaneously and defined to be unrelated.

## REFERENCES

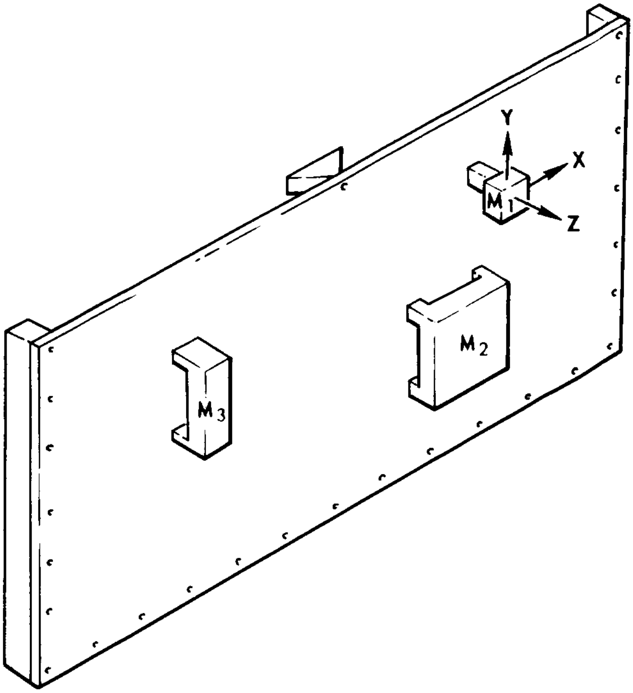

1. Bisplinghoff, Raymond L., et.al "Aeroelasticity" (Addison-Wesley, 1957)

2. Aseltine, John A., "Transform Methods of Linear System Analysis" (McGraw-Hill, 1958)

3. Denke, P. H., "A Matrix Method of Structural Analysis"

4. Denke, P. H., "A General Digital Computer Analysis of Statically Indetermined Structures"

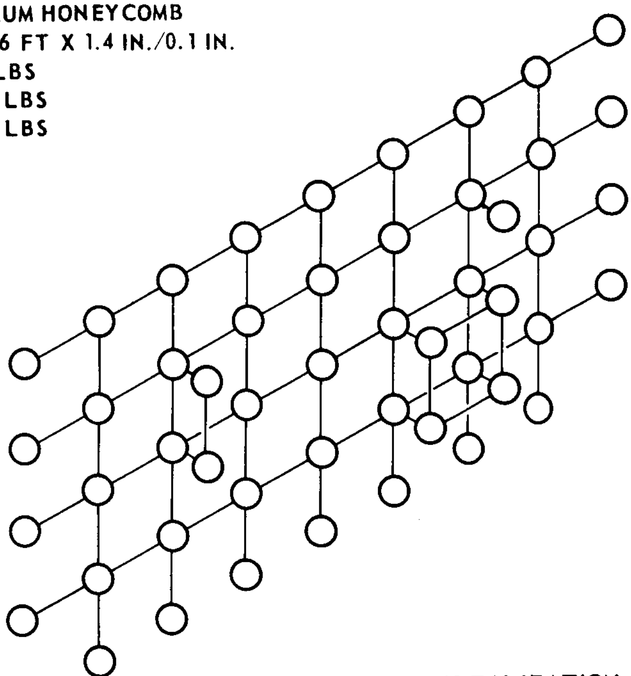
## APPENDIX 1




STRUCTURAL PLATE WITH MOUNTED COMPONENTS

**FIGURE 1**

PHYSICAL DATA
ALUMINUM HONEYCOMB
3 FT X 6 FT X 1.4 IN./0.1 IN.
$M_1$ = 8 LBS
$M_2$ = 24 LBS
$M_3$ = 20 LBS



LUMPED PARAMETER IDEALIZATION

**FIGURE 2**

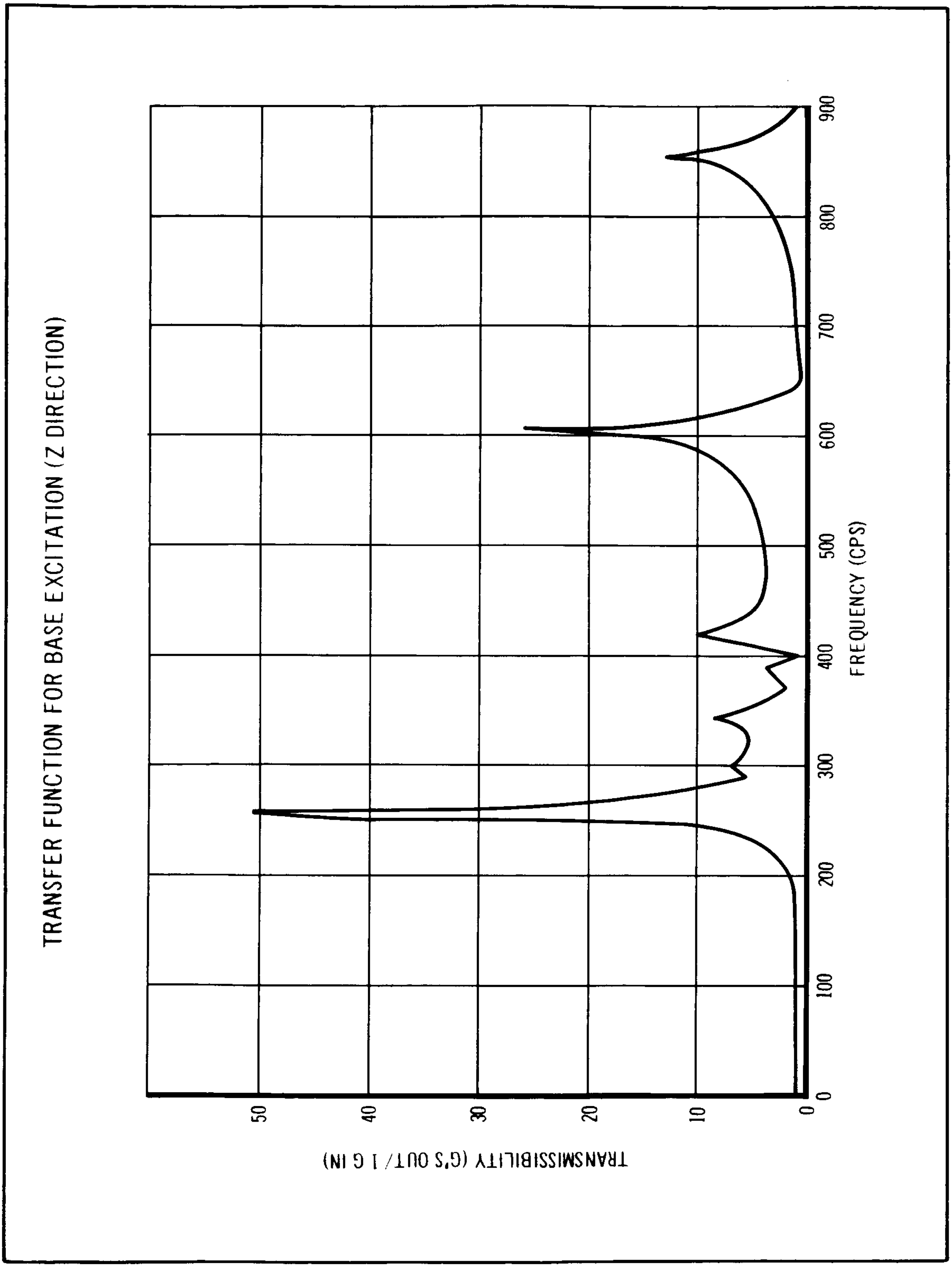
## APPENDIX 2

## TABLE 1

## NATURAL FREQUENCIES

| MODE NUMBER | FREQUENCY - CPS |
|---|---|
| 1 | 257.034 |
| 2 | 267.266 |
| 3 | 292.780 |
| 4 | 348.127 |
| 5 | 393.383 |
| 6 | 418.646 |
| 7 | 483.426 |
| 8 | 605.785 |
| 9 | 854.014 |

## APPENDIX 3



TRANSFER FUNCTION FOR BASE EXCITATION (Z DIRECTION)
TRANSMISSIBILITY (G'S OUT / 1 G IN)
0
10
20
30
40
50
FREQUENCY (CPS)
0
100
200
300
400
500
600
700
800
900


**CHART 1.**

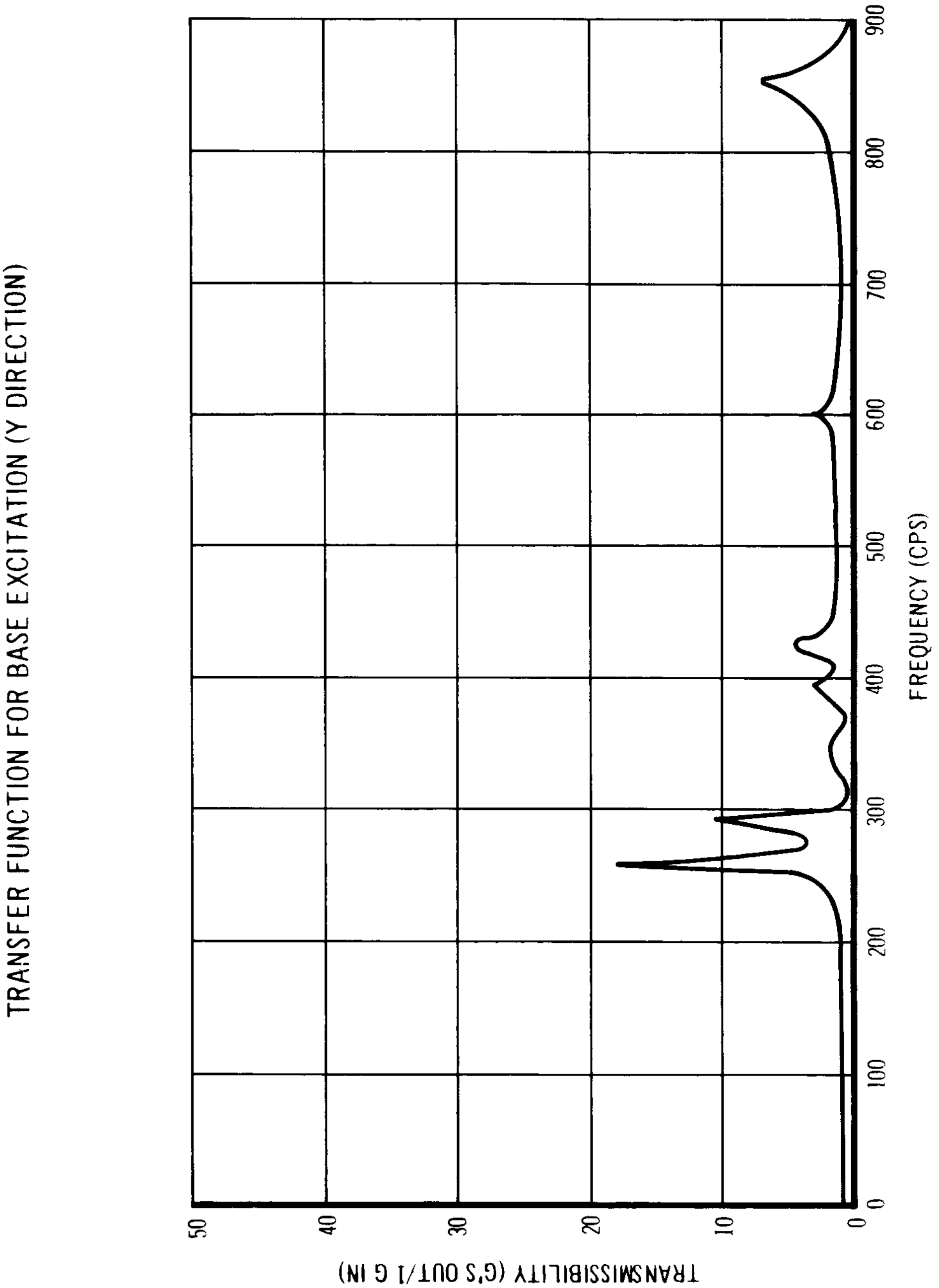

TRANSFER FUNCTION FOR BASE EXCITATION (Y DIRECTION)
TRANSMISSIBILITY (G'S OUT/1 G IN)
0
10
20
30
40
50
FREQUENCY (CPS)
0
100
200
300
400
500
600
700
800
900

CHART 2.

TRANSFER FUNCTION FOR BASE EXCITATION (X DIRECTION)

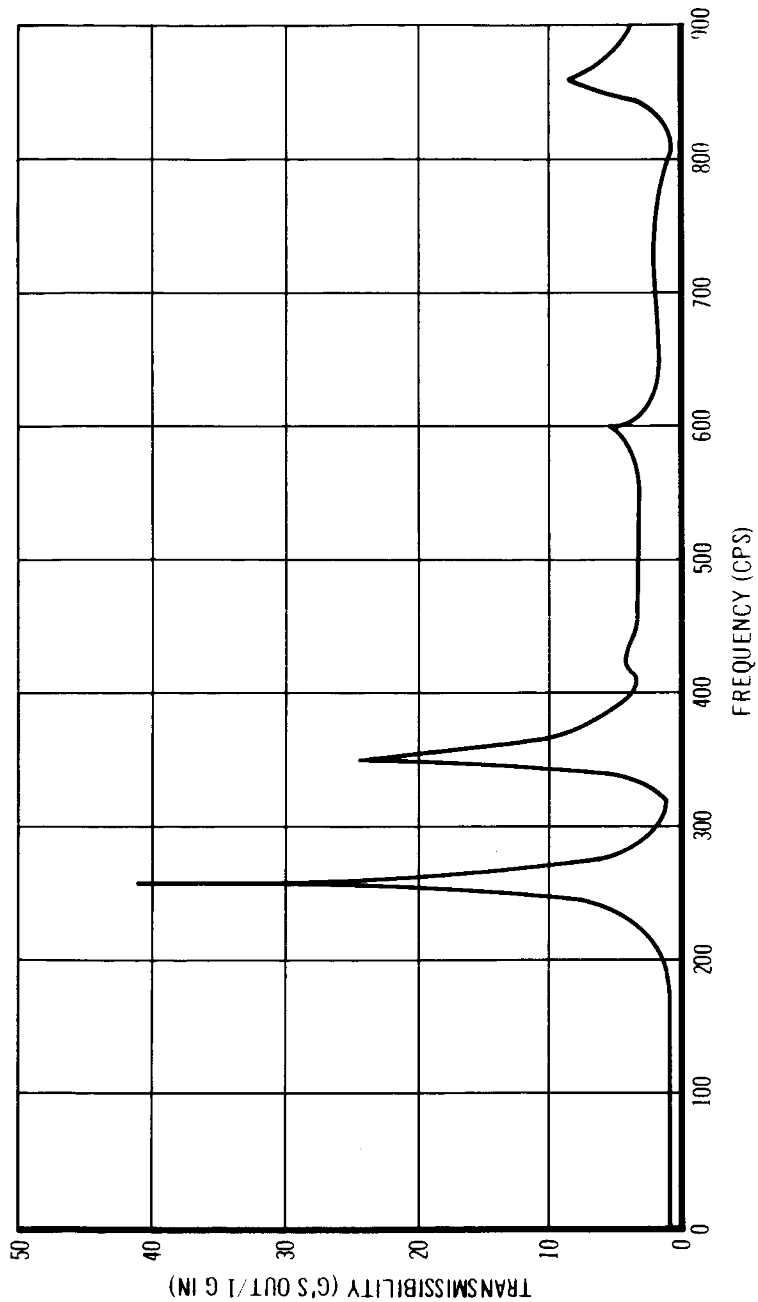
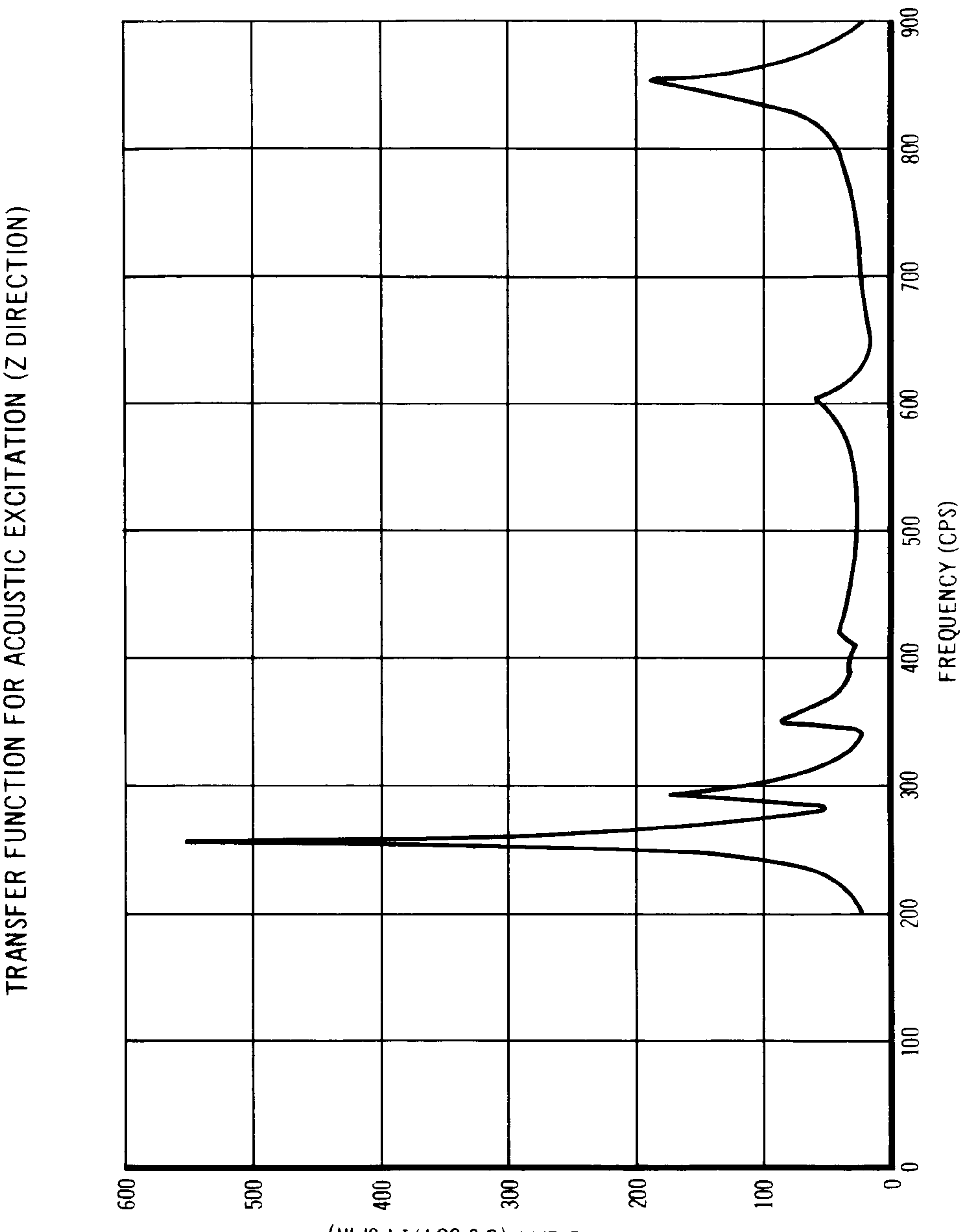



CHART 3.



TRANSFER FUNCTION FOR ACOUSTIC EXCITATION (Z DIRECTION)
TRANSMISSIBILITY (G'S OUT/1 PSI IN)
FREQUENCY (CPS)
0
100
200
300
400
500
600
700
800
900


**CHART 4.**

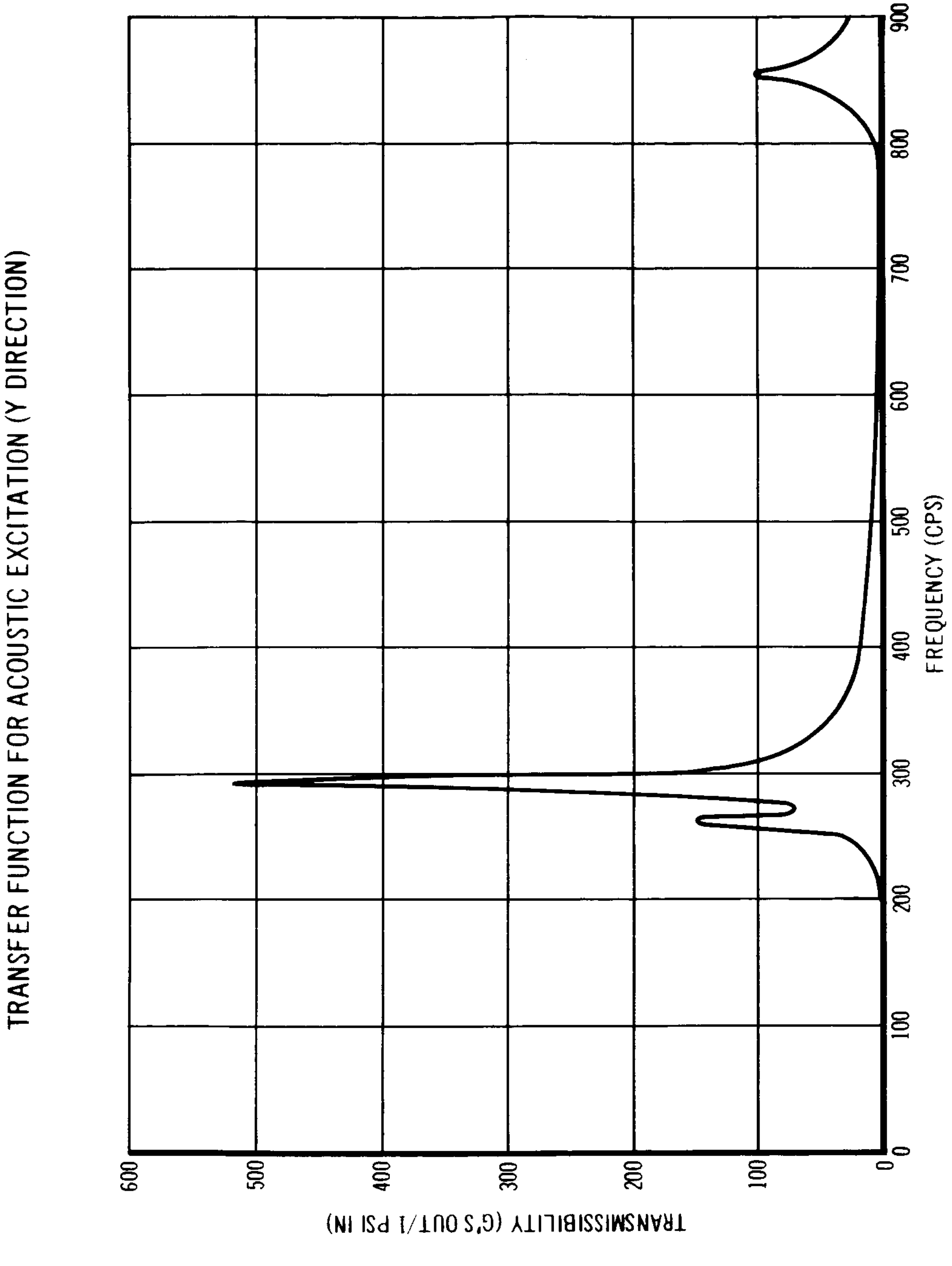

TRANSFER FUNCTION FOR ACOUSTIC EXCITATION (Y DIRECTION)
FREQUENCY (CPS)
0
100
200
300
400
500
600
700
800
900
TRANSMISSIBILITY (G'S OUT/1 PSI IN)
0
100
200
300
400
500
600

CHART 5.

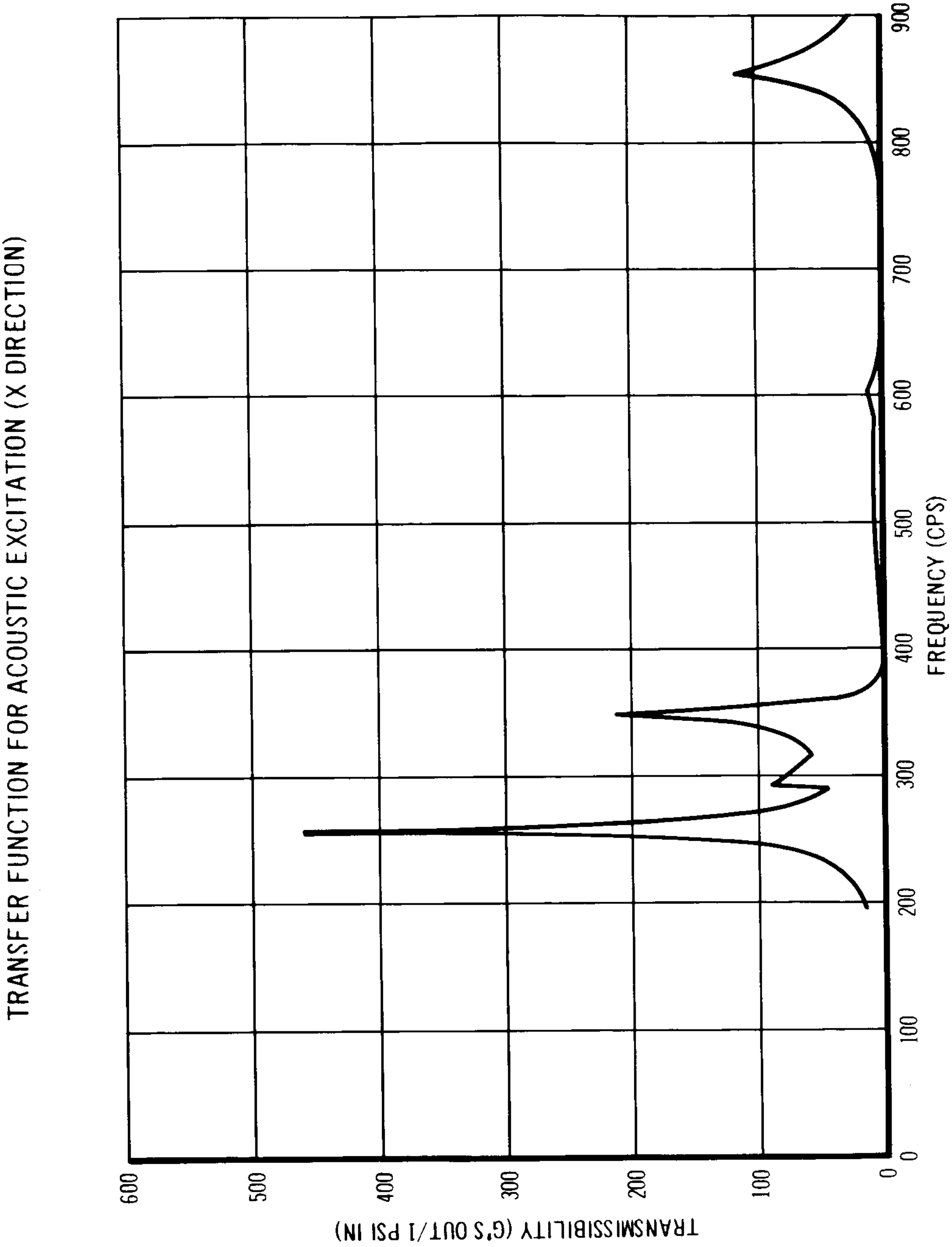

TRANSFER FUNCTION FOR ACOUSTIC EXCITATION (X DIRECTION)
TRANSMISSIBILITY (G'S OUT/1 PSI IN)
0
100
200
300
400
500
600
FREQUENCY (CPS)
0
100
200
300
400
500
600
700
800
900

CHART 6.

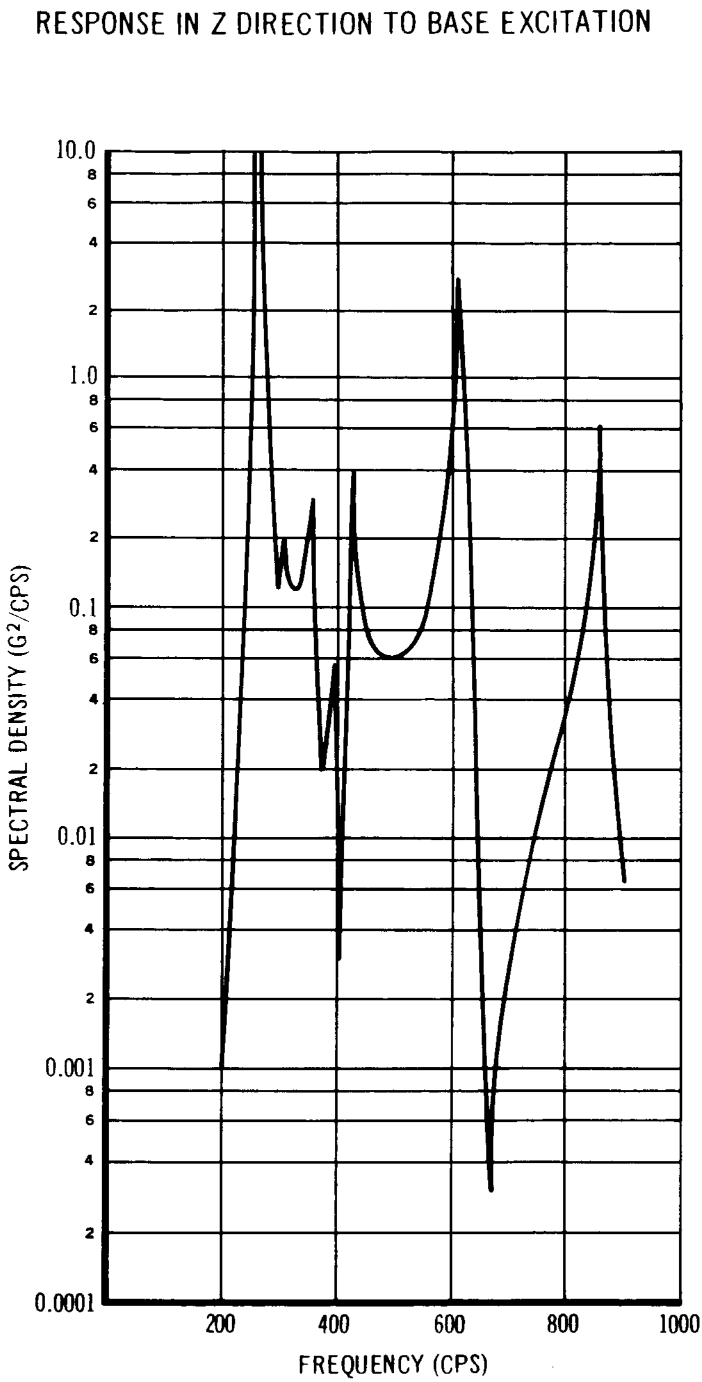

RESPONSE IN Z DIRECTION TO BASE EXCITATION
SPECTRAL DENSITY (G²/CPS)
10.0
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
0.01
8
6
4
2
0.001
8
6
4
2
0.0001
200
400
600
800
1000
FREQUENCY (CPS)

CHART 7.

RESPONSE IN Y DIRECTION TO BASE EXCITATION

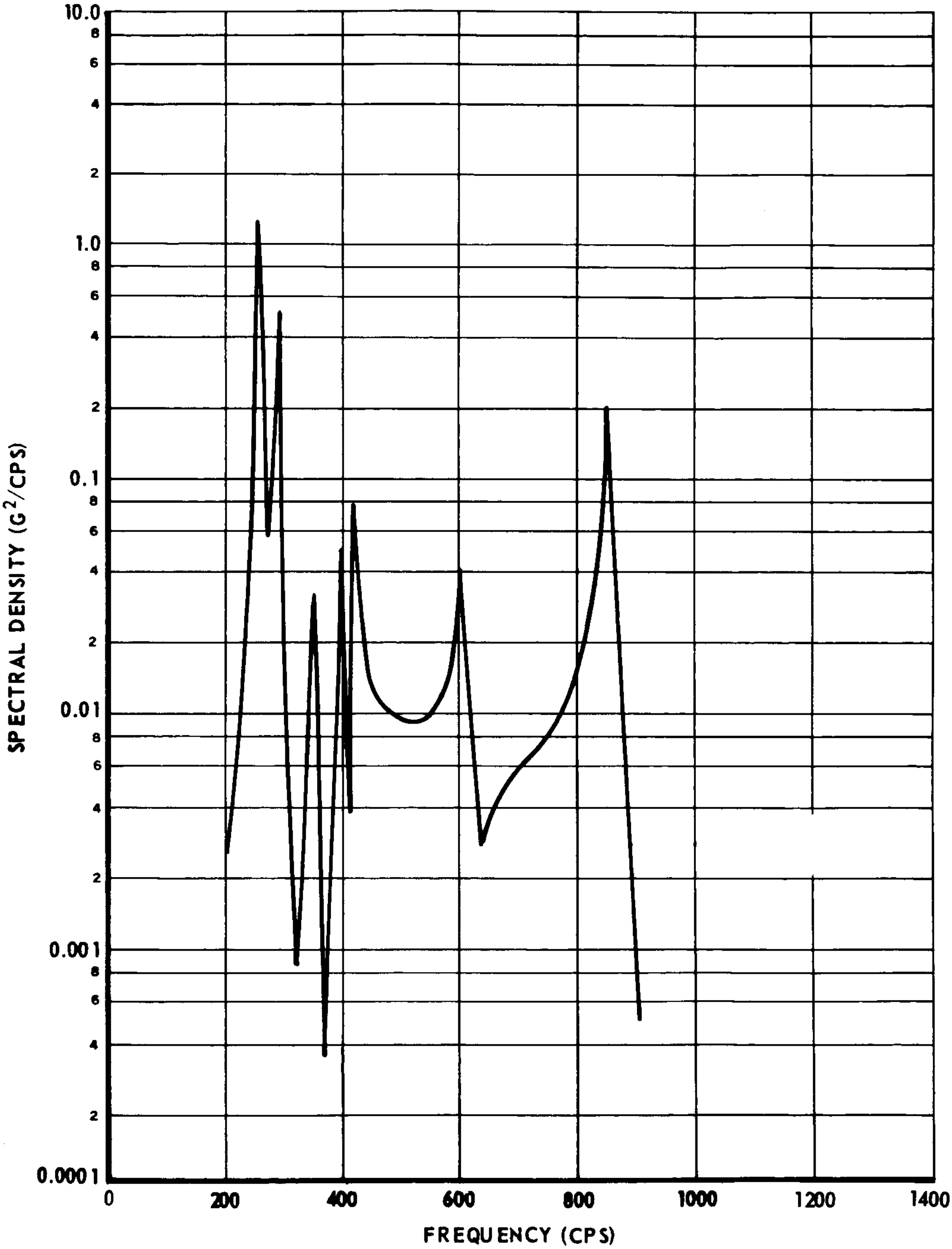



CHART 8.

RESPONSE IN X DIRECTION TO BASE EXCITATION

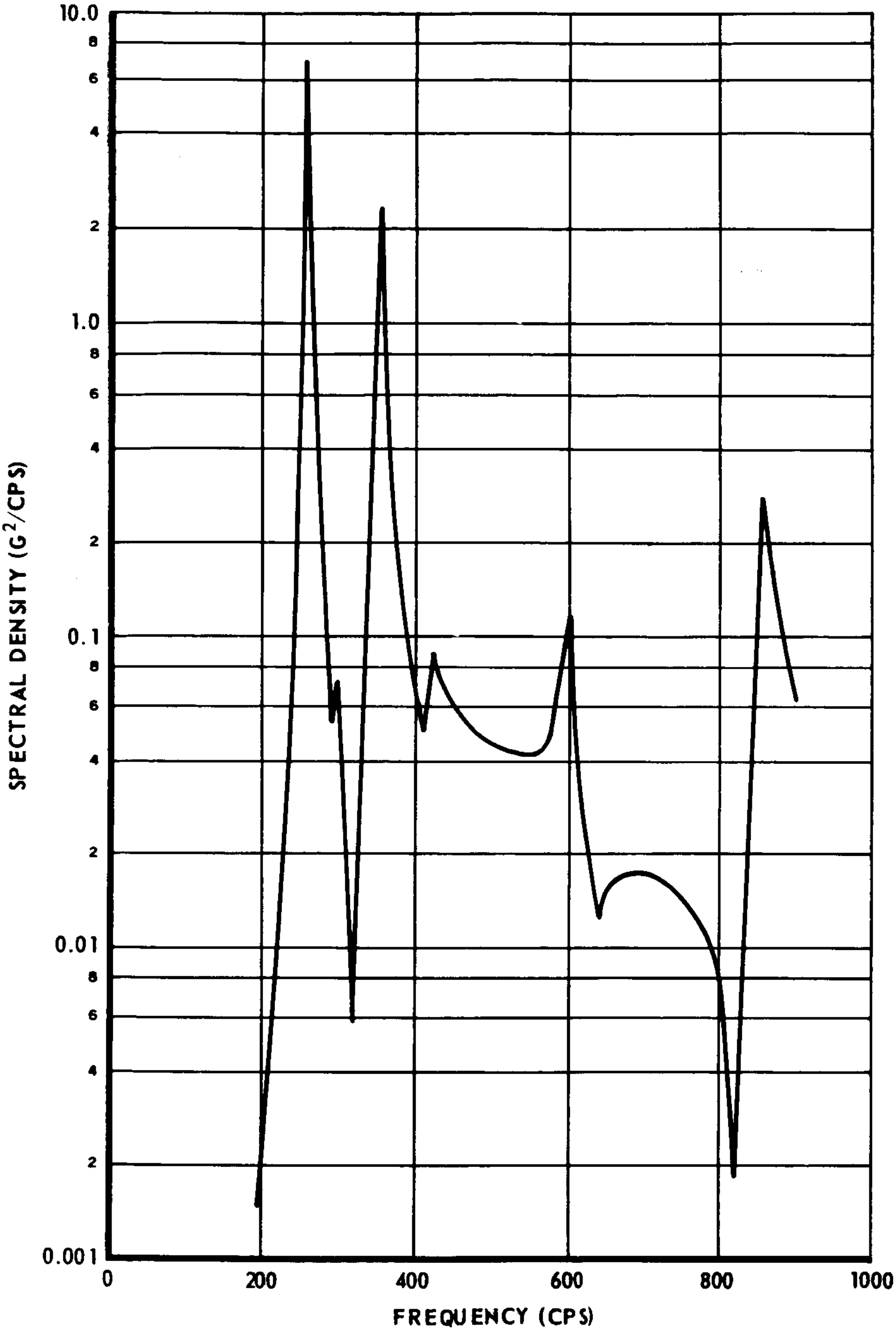



CHART 9.

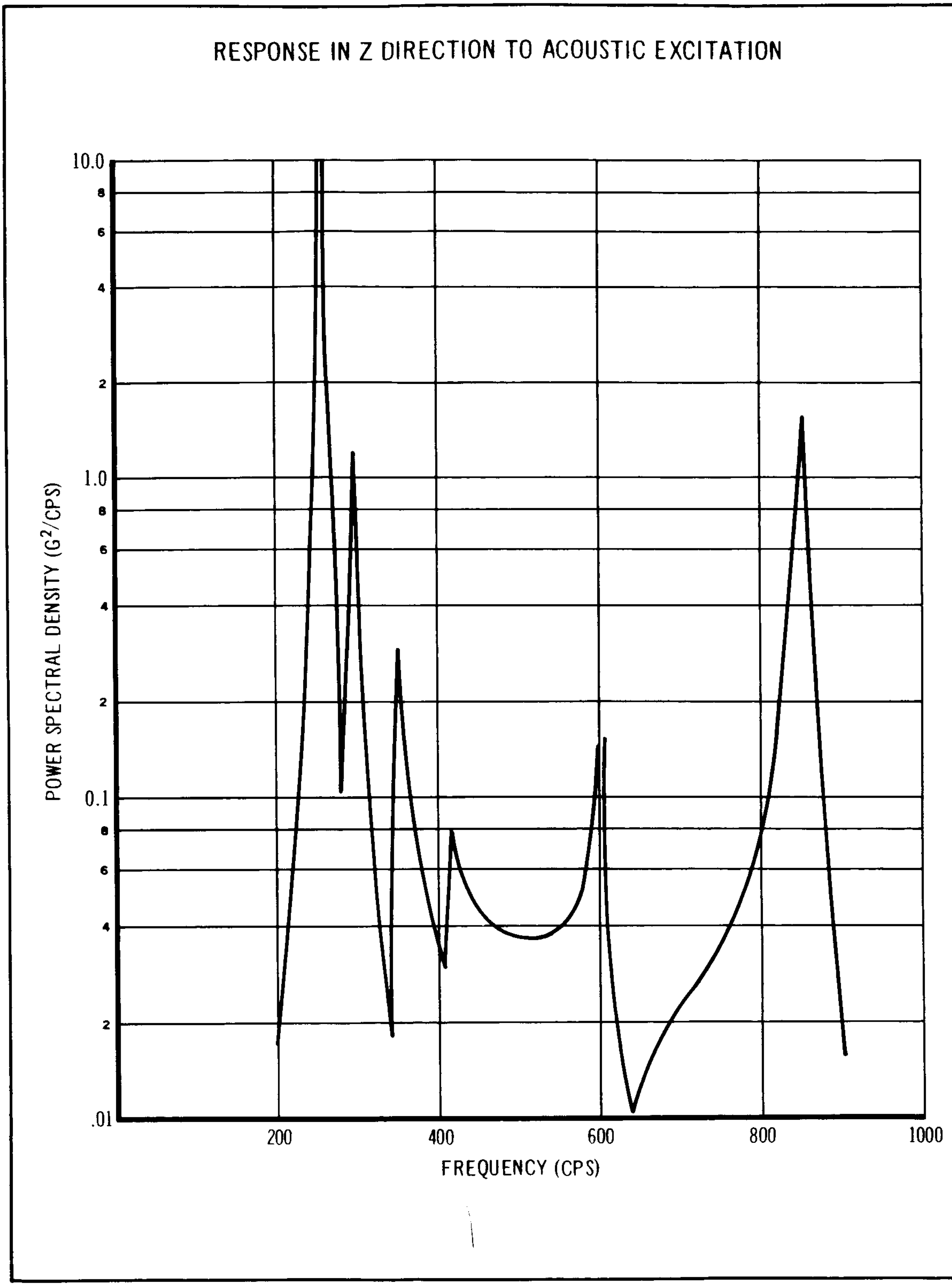

RESPONSE IN Z DIRECTION TO ACOUSTIC EXCITATION
POWER SPECTRAL DENSITY ($G^2$/CPS)
10.0
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
.01
200
400
600
800
1000
FREQUENCY (CPS)

CHART 10.

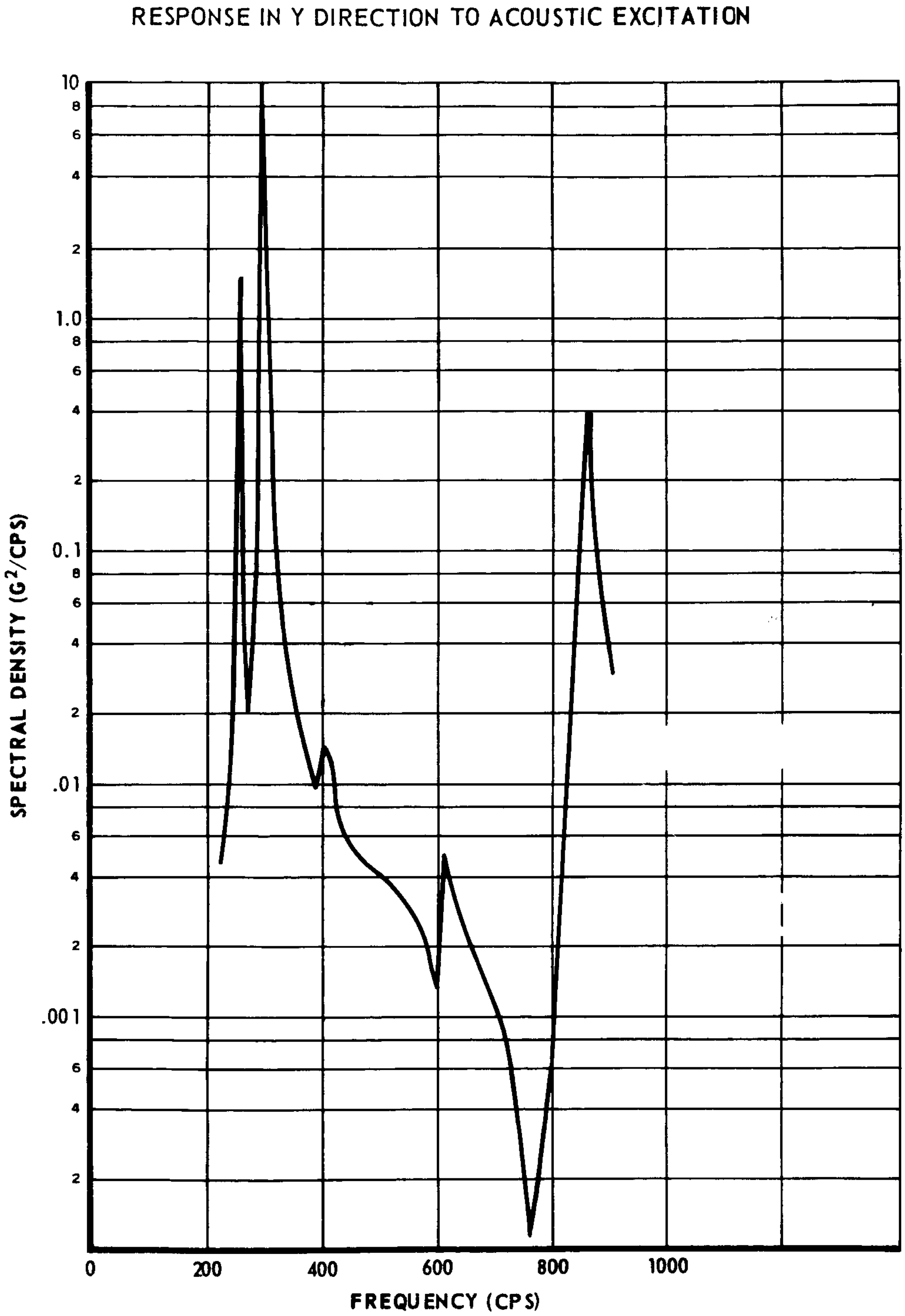

RESPONSE IN Y DIRECTION TO ACOUSTIC EXCITATION
10
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
.01
6
4
2
.001
6
4
2
SPECTRAL DENSITY ($G^2$/CPS)
0
200
400
600
800
1000
FREQUENCY (CPS)

CHART 11.

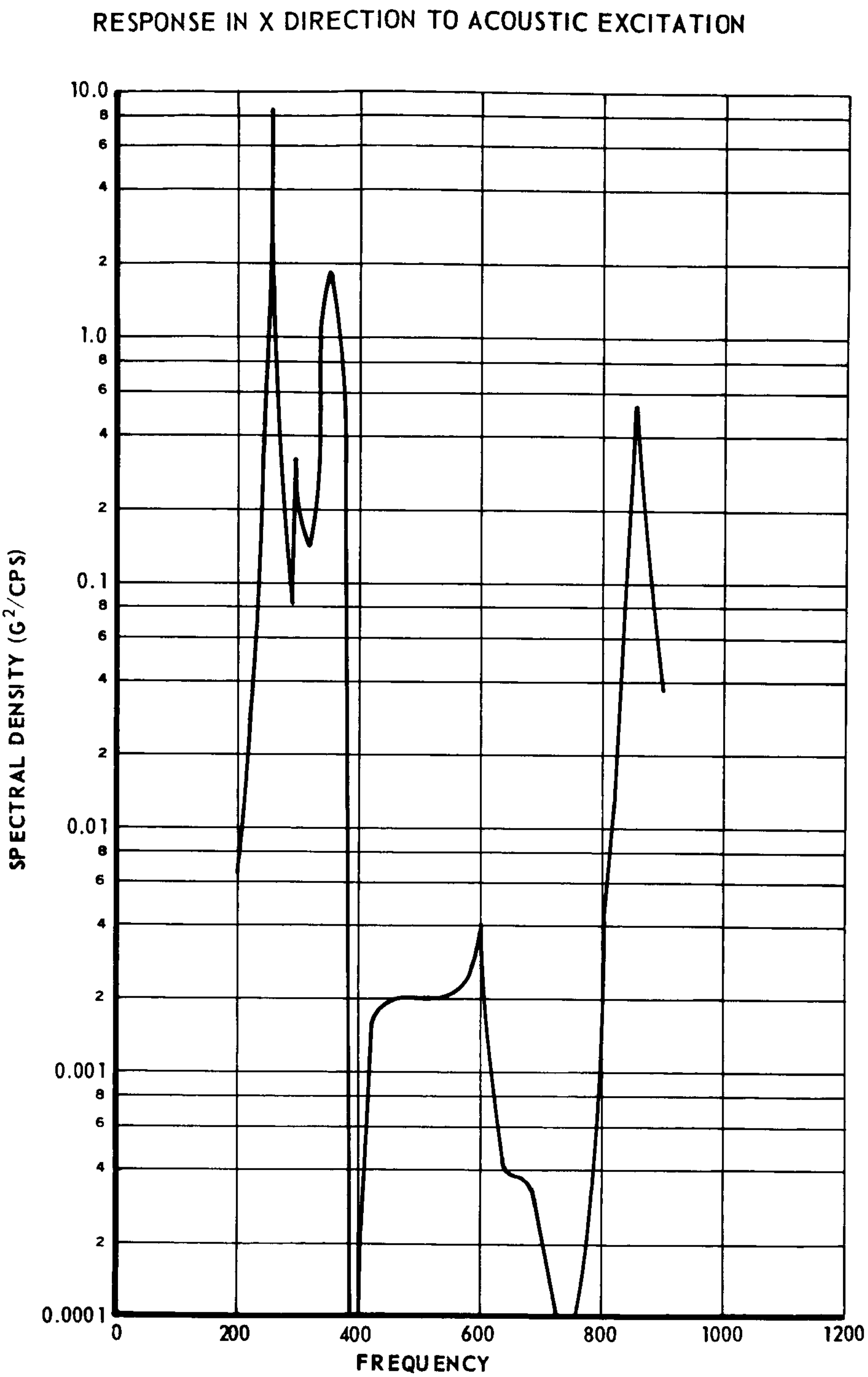

RESPONSE IN X DIRECTION TO ACOUSTIC EXCITATION
SPECTRAL DENSITY ($G^2$/CPS)
10.0
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
0.01
8
6
4
2
0.001
8
6
4
2
0.0001
0
200
400
600
800
1000
1200
FREQUENCY

CHART 12.

# RESPONSE IN Z DIRECTION TO COMBINED EXCITATION

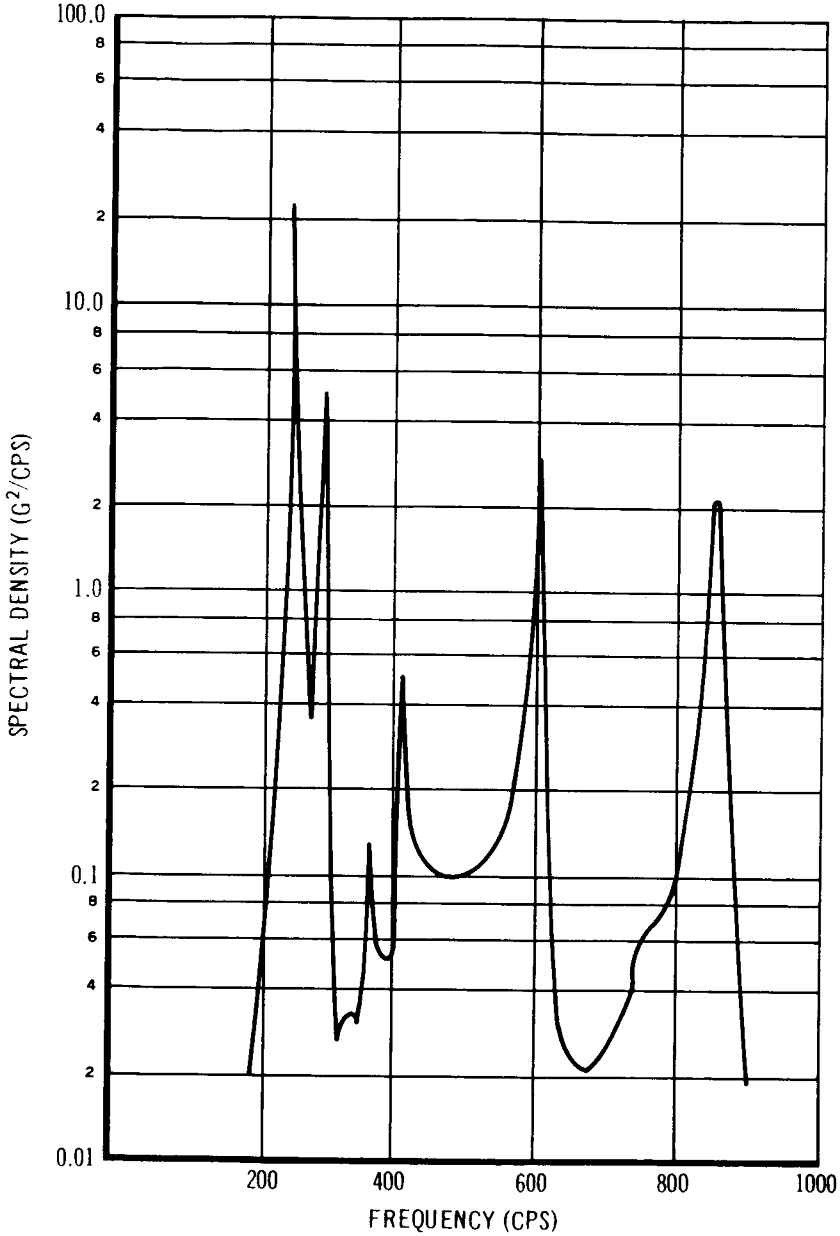
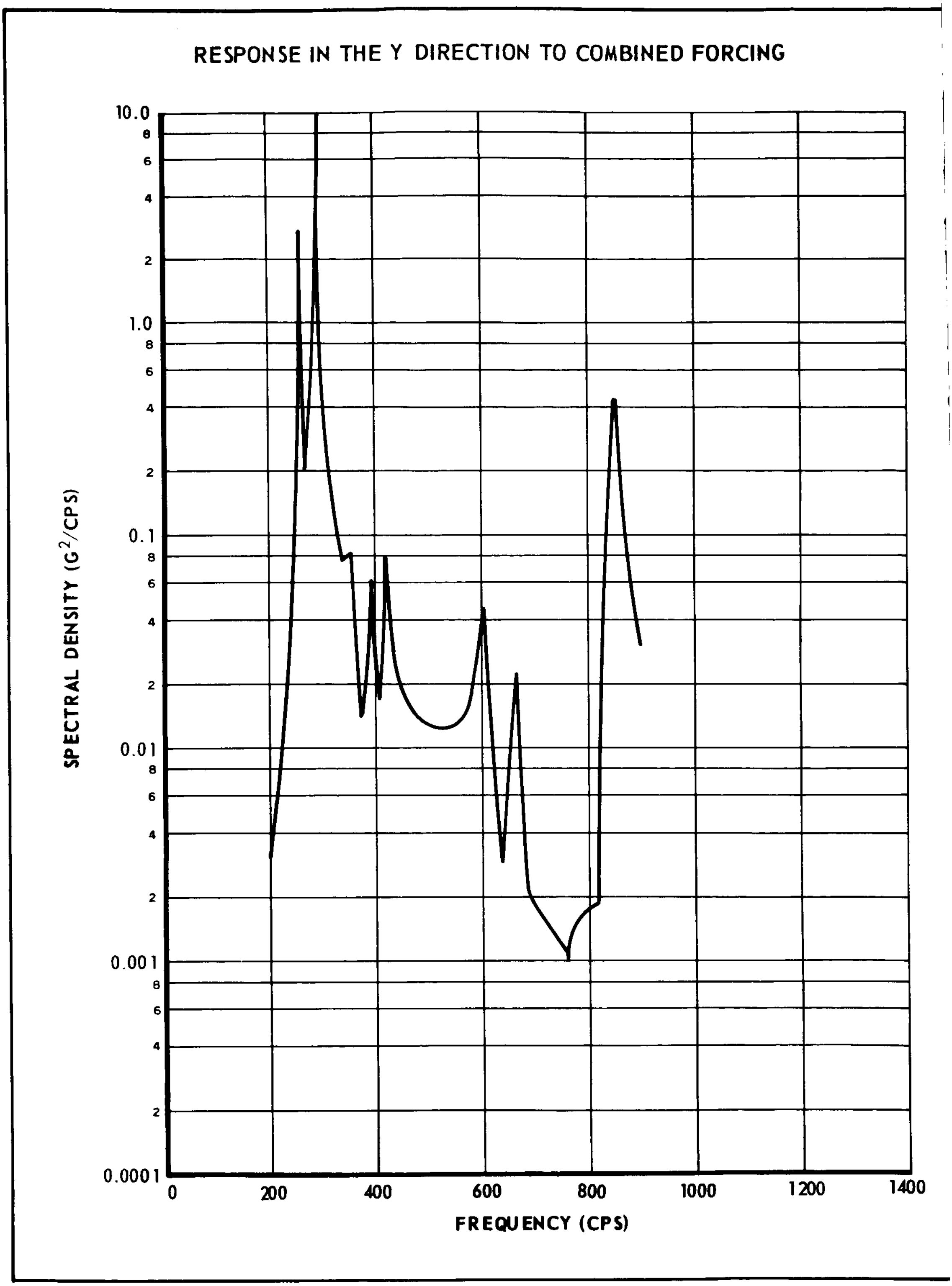



**CHART 13.**



RESPONSE IN THE Y DIRECTION TO COMBINED FORCING
SPECTRAL DENSITY ($G^2$/CPS)
10.0
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
0.01
8
6
4
2
0.001
8
6
4
2
0.0001
0
200
400
600
800
1000
1200
1400
FREQUENCY (CPS)


CHART 14.

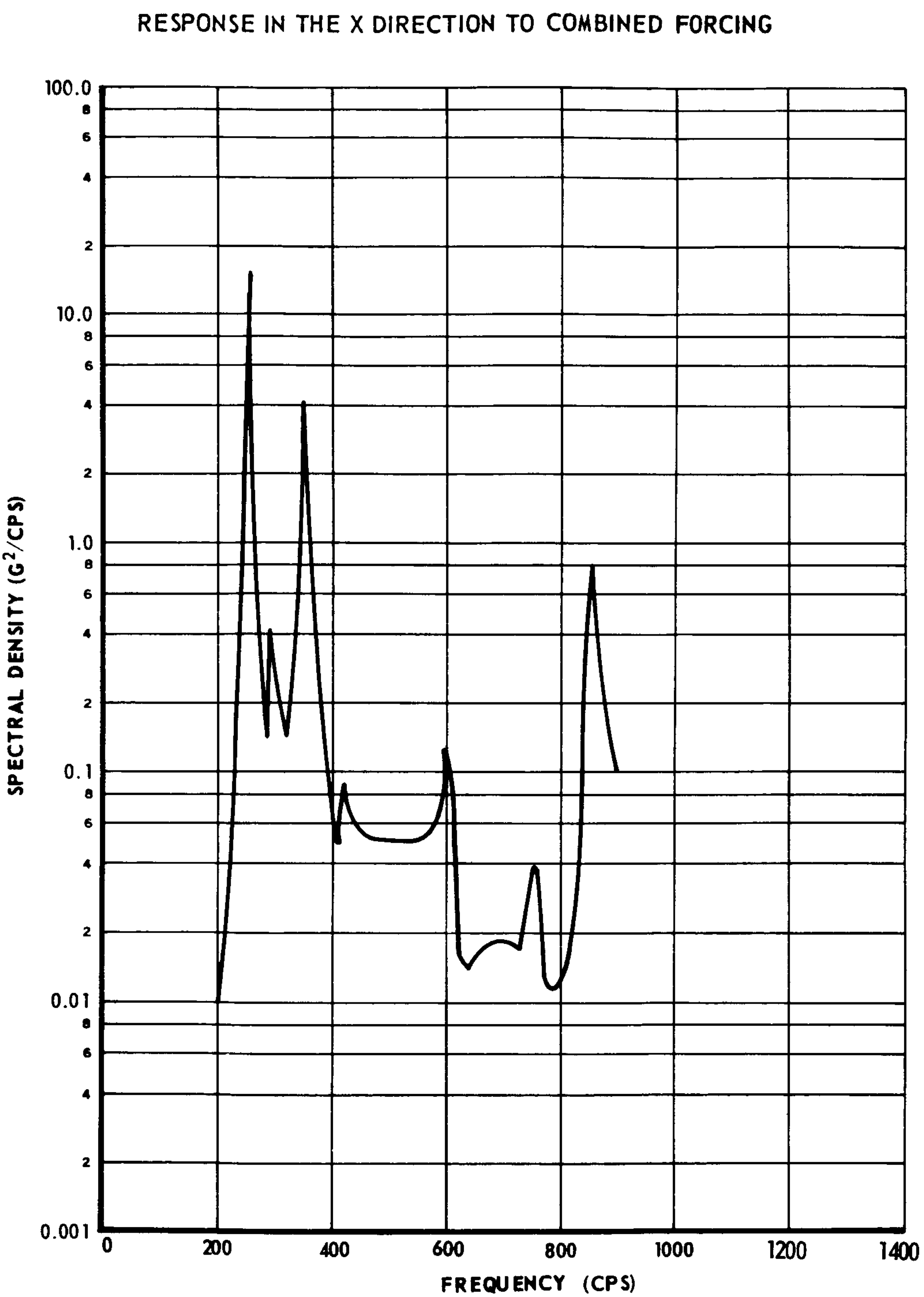

RESPONSE IN THE X DIRECTION TO COMBINED FORCING
100.0
8
6
4
2
10.0
8
6
4
2
1.0
8
6
4
2
0.1
8
6
4
2
0.01
8
6
4
2
0.001
SPECTRAL DENSITY ($G^2$/CPS)
0
200
400
600
800
1000
1200
1400
FREQUENCY (CPS)

CHART 15.